Relaxation Oscillations of the Synchrotron Motion Caused by Narrow-Band Impedance

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1 Outline

- SPEAR parameters
- Motivations
- Experimental data to characterize phenomenon
  - Spectrum analyzer
  - Streak camera
- Simulations to increase understanding
- Analytical model that explains this behavior

2 SPEAR Parameters

3 GeV e^- storage ring dedicated to synchrotron radiation

- ultra-relativistic ($\beta \cong 1$) $\Rightarrow$ no transition
- space charge effects negligible
- signi• cant synchrotron radiation and associated damping

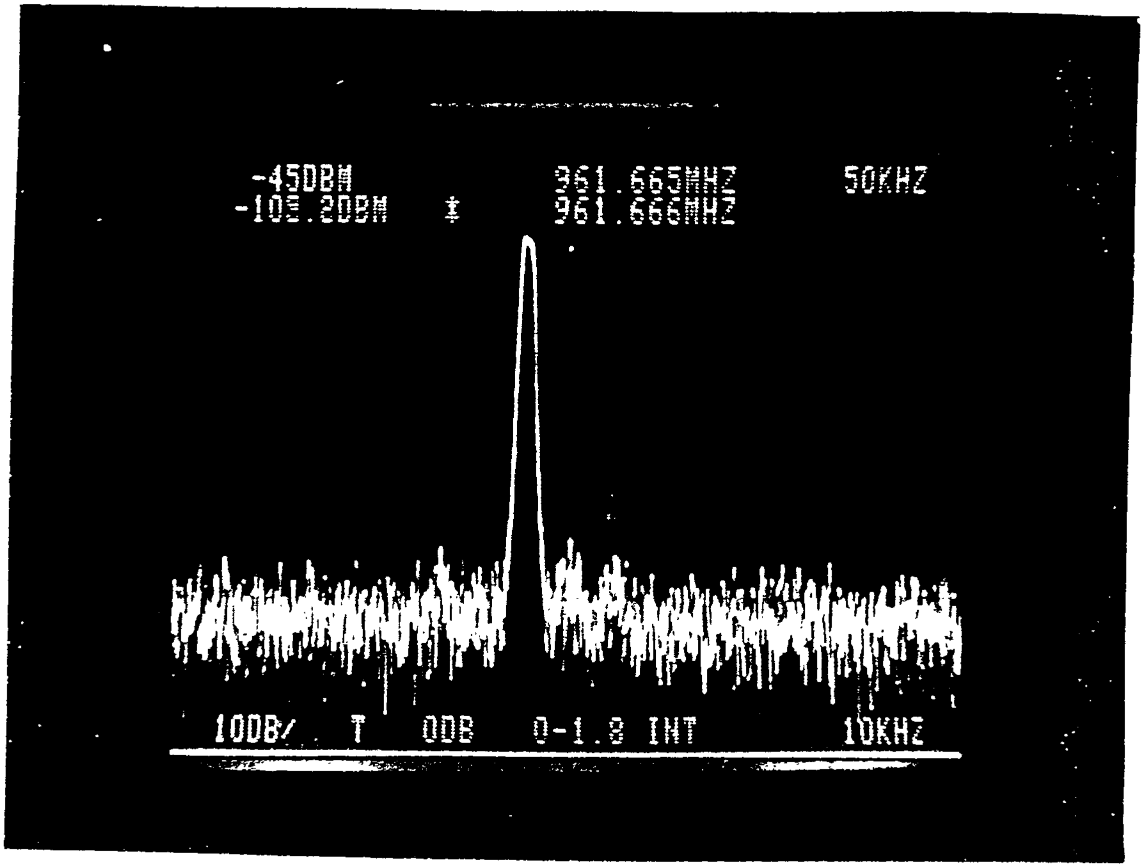
Energy	2.3GeV	<u>HOM studied</u>	= Fundamental
Natural Damping Time	10 ms	$f_{res} = f_{rf}$	= 358.533MHz
Measured Damping Time	5 ms (@2mA)		
		R_s = 10 MΩ	
Bunch spectrum (σ_τ^{-1})	2.8 GHz	57 ps	1/49 RF bucket
$f_{rf} = f_{HOM}$	358.533 MHz	2.8 ns	1/280 turn
f_o	1.28 MHz	0.78 μs	1 turn
Damping of HOM resonance	56 kHz	17.8 μs	23 turns
f_{so}	28.4 kHz	35 μs	45 turns
Relaxation oscillation	<100Hz	>10 ms	>12800 turns

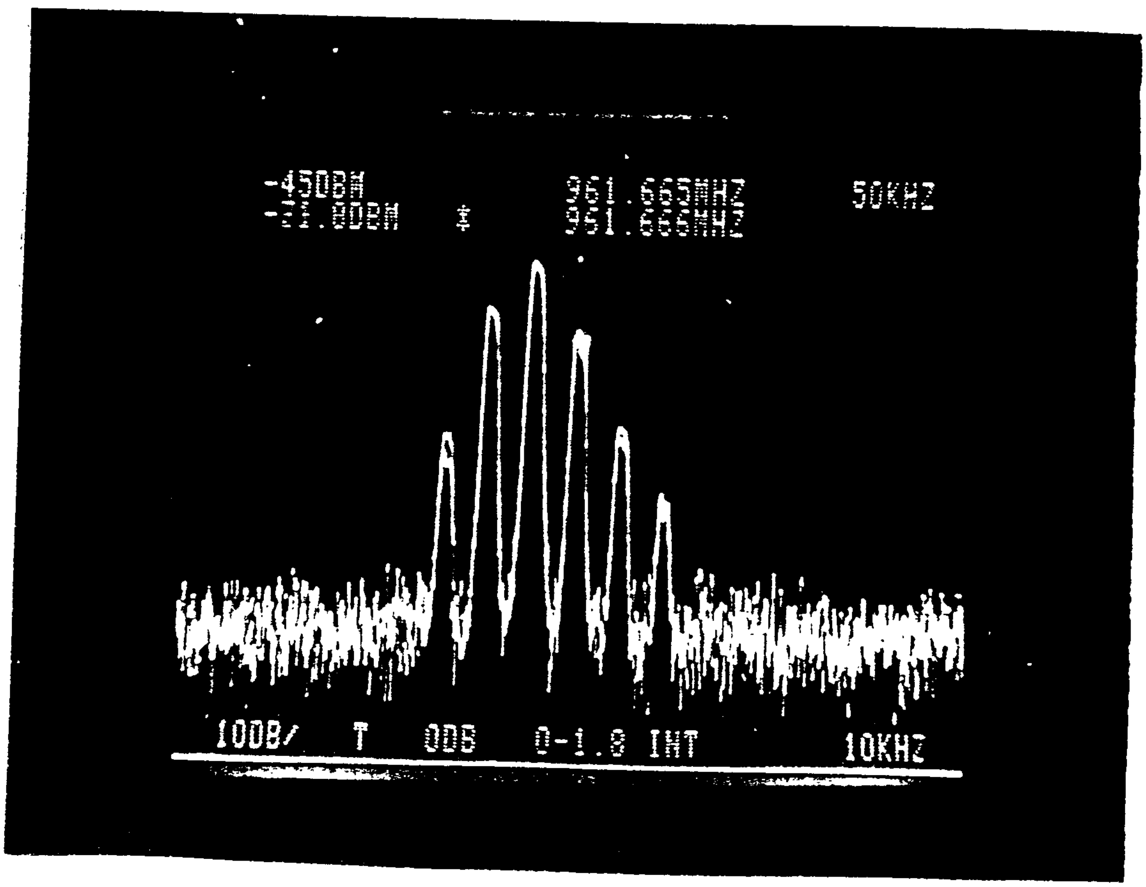
3 Motivation

- SPEAR has two 5-cell PEP-I/LEP style RF cavities
  - One cavity active, one cavity passive in normal operation
  - Two moveable tuners in each 5-cell cavity
- Work done to improve stability of operating point with respect to cavity HOMs
  - No HOM dampers on cavities
  - No longitudinal or transverse feedback systems
  - No temperature regulation on cavities
- Characterized HOMs vs tuner position

4 Observations

- Longitudinal oscillations saturate
- Envelope, itself, oscillates at very low frequency (three orders of magnitude smaller than ω_s)
- Previously experimentally observed and reported
  - PhotonFactory [Yamazaki 1983]
  - Surf 2 [Rakowsky 1985]
  - Elettra [Wrulich 1996]
- Previously studied theoretically
  - Suzuki and Yokoya [1982]
  - Krinsky [1985]
  - Nagaoka [1996]
- Characterized this behavior on largest impedance available, the fundamental mode of the idle cavity, using signals from a pickup within the cavity


-45DBM
-103.2DBM
961.665MHZ
961.666MHZ
50KHZ
10DB/
T
0DB
0-1.8 INT
10KHZ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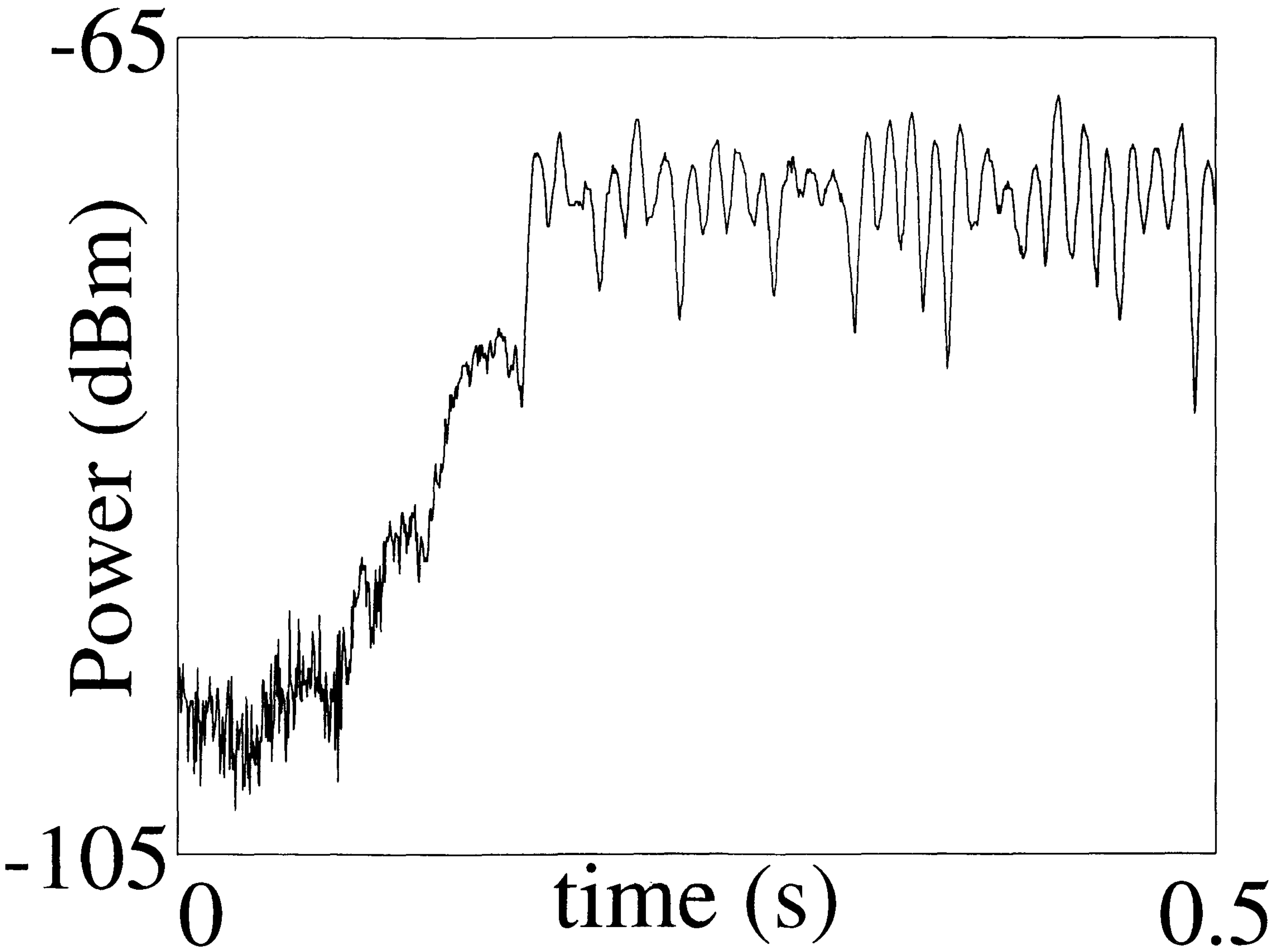


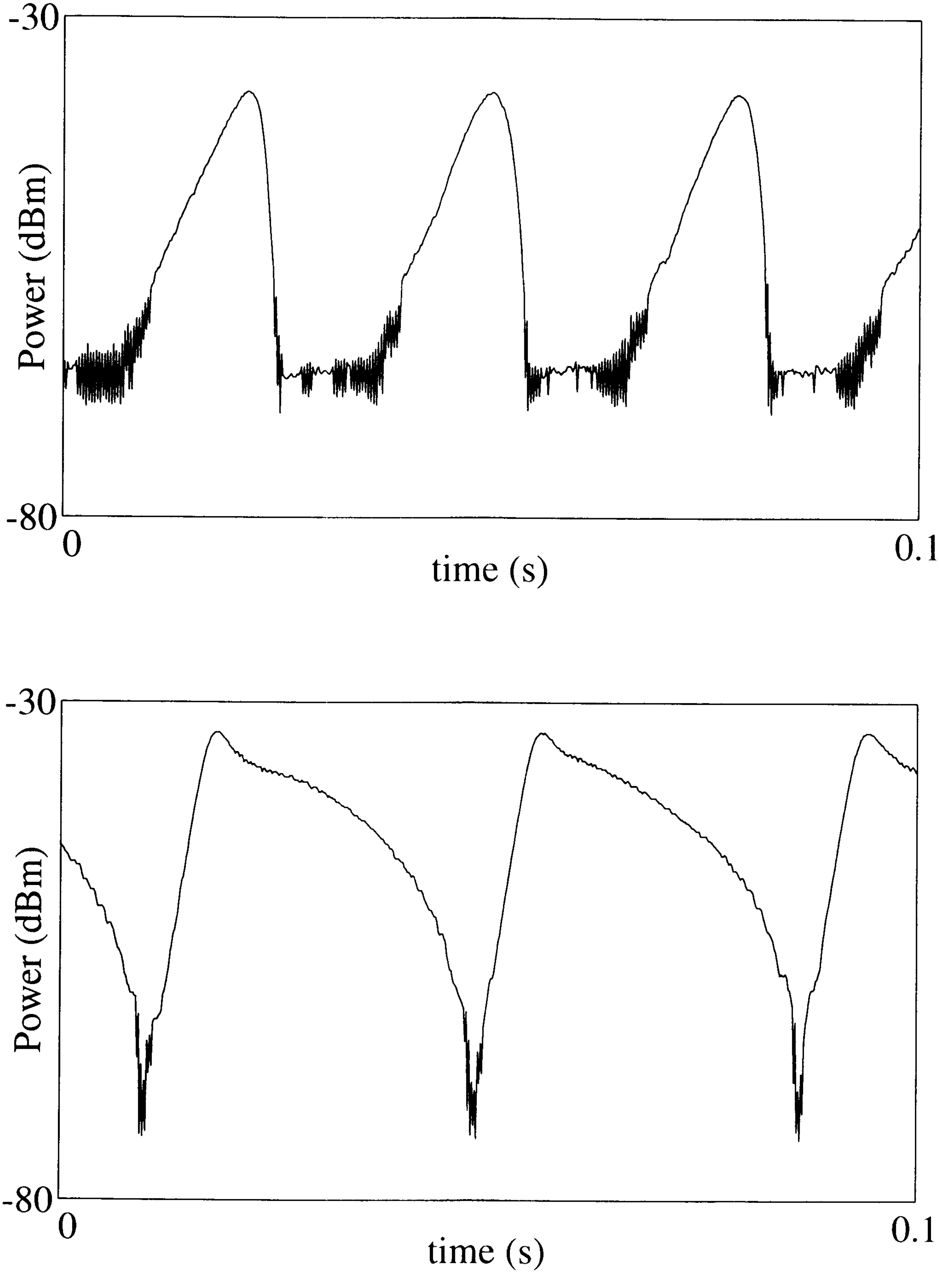
-65
-105
Power (dBm)
0
time (s)
0.5

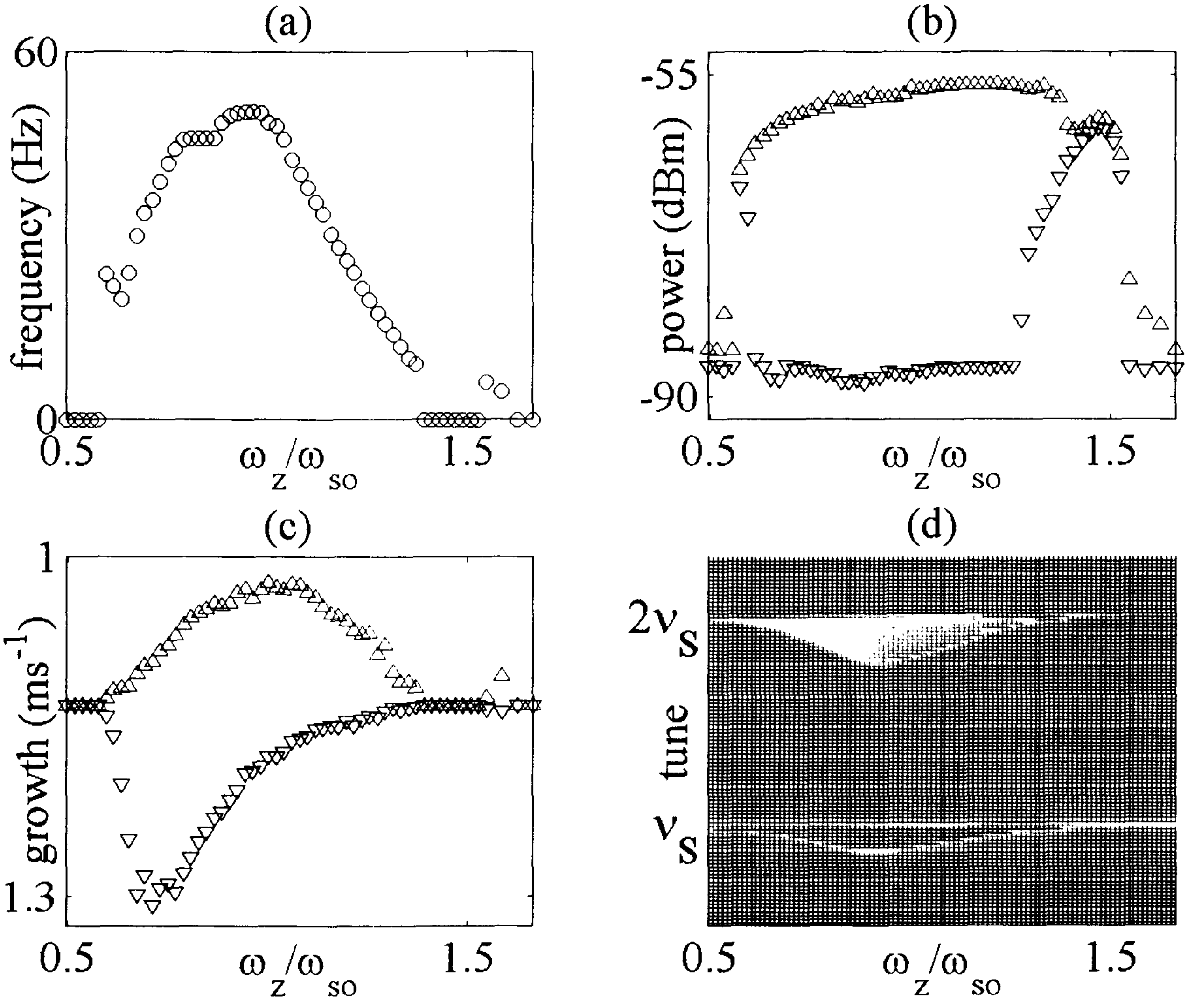
5 Spectrum Analyzer Results

- Low oscillation frequency < 100 Hz $\quad (\sim \tau^{-1}_{radiation\, damping})$
- Extends almost over entire region of instability
- Symmetry of growth time follows $R(\omega_{HOM}) = R\,(p\omega_0 + \omega_z)$
- Asymmetry of damping
  - Complex damping mechanism
  - Explains frequency asymmetry
- Broadening of synchrotron frequency line:
  - $\Delta f_s \sim -15\%$

The oscillation amplitude initially grows toward an attractor at ∞. Then the dynamics of the system change so that the motion evolves toward another attractor. We call this cycle a **relaxation oscillation**.

These data give information only about the dipole moment of the bunch. We want to see its internal structure as well power of upper ν_s sideband vs time





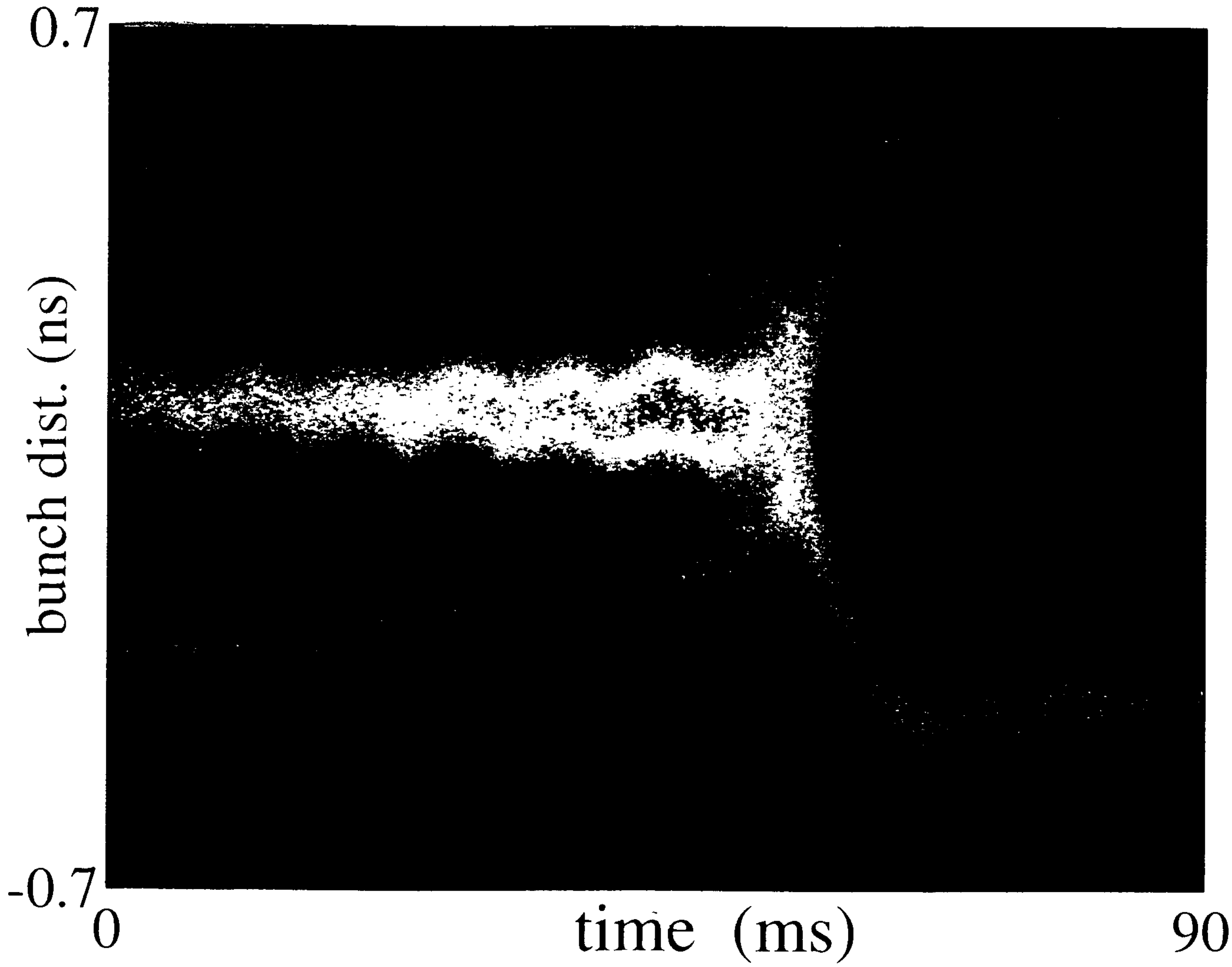
(a)
60
0
frequency (Hz)
0.5
1.5
ω_z/ω_so
(b)
-55
-90
power (dBm)
0.5
1.5
ω_z/ω_so
(c)
1
-1.3
growth (ms⁻¹)
0.5
1.5
ω_z/ω_so
(d)
2ν_s
ν_s
tune
0.5
1.5
ω_z/ω_so

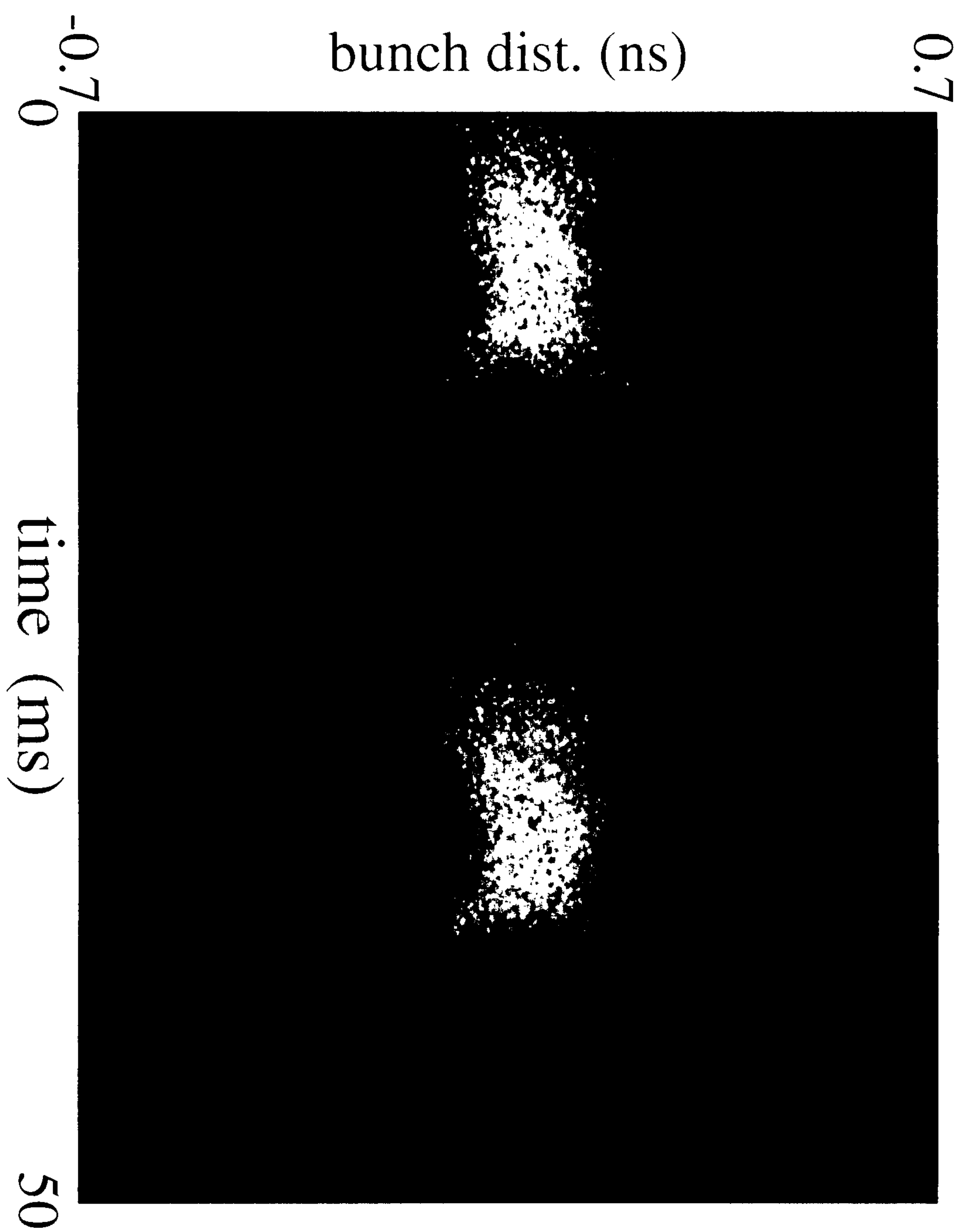
6 Streak Camera

- Since the bunch is too short to examine its time distribution electronically, we used a streak camera to optically view this dimension.
- We were assisted by colleagues J. Hinkson and J. Byrd (LBNL), A. Fisher (SLAC) and A. Lumpkin (APS) in obtaining the beautiful pictures
- Dual sweep streak camera
  - Synchrotron radiation emitted by bunch is image of longitudinal e^- distribution
  - Camera converts γ's to e^-'s in camera photocathode
  - Rotates longitudinal distribution of e^- to vertical distribution
  - Sweeps slowly in horizontal direction to image different bunches
  - Shows bunch envelope over long times on slow sweeps
  - Synchronized with f_{RF}, it shows bunch by bunch distributions on fast sweeps

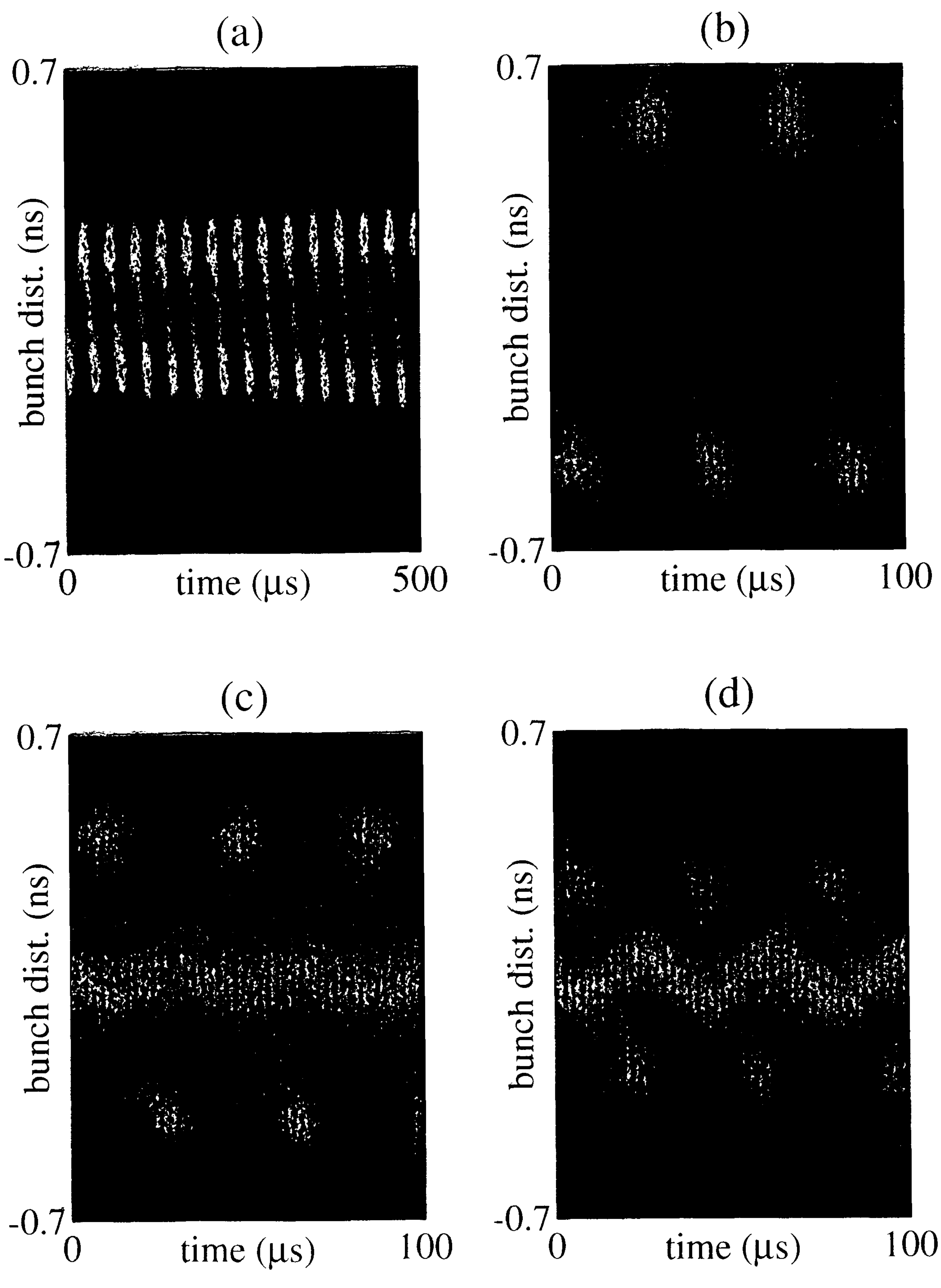
7 Results from Streak Camera Images

- Correlates, as expected, with data from spectrum analyzer
- Stays con• ned as a single macroparticle in intial phase of relaxation cycle
- Loss of intensity during growth
  - Particles start to escape from the bunch and are seen distributed over time
- Large amplitude of oscillations ($\pm\pi/2$) i.e. 1/2 RF bucket size
  - Pendulum frequency decreases quadratically with amplitude
  - Spectrum analyzer data showed this 15% frequency shift over the relaxation cycle
- Attractor at • nite amplitude
- In most cases, images show bunch collapses to center and begins new cycle
- In the particular case of $\omega_z > \omega_s$ and a very slow damping rate, a 2^{nd} attractor appears at the center
  - $\sim \pi$ out of phase with main body (or 'initial attractor')
  - starts growing while the 'initial attractor' is still damping


0.7
-0.7
bunch dist. (ns)
0
time (ms)
90



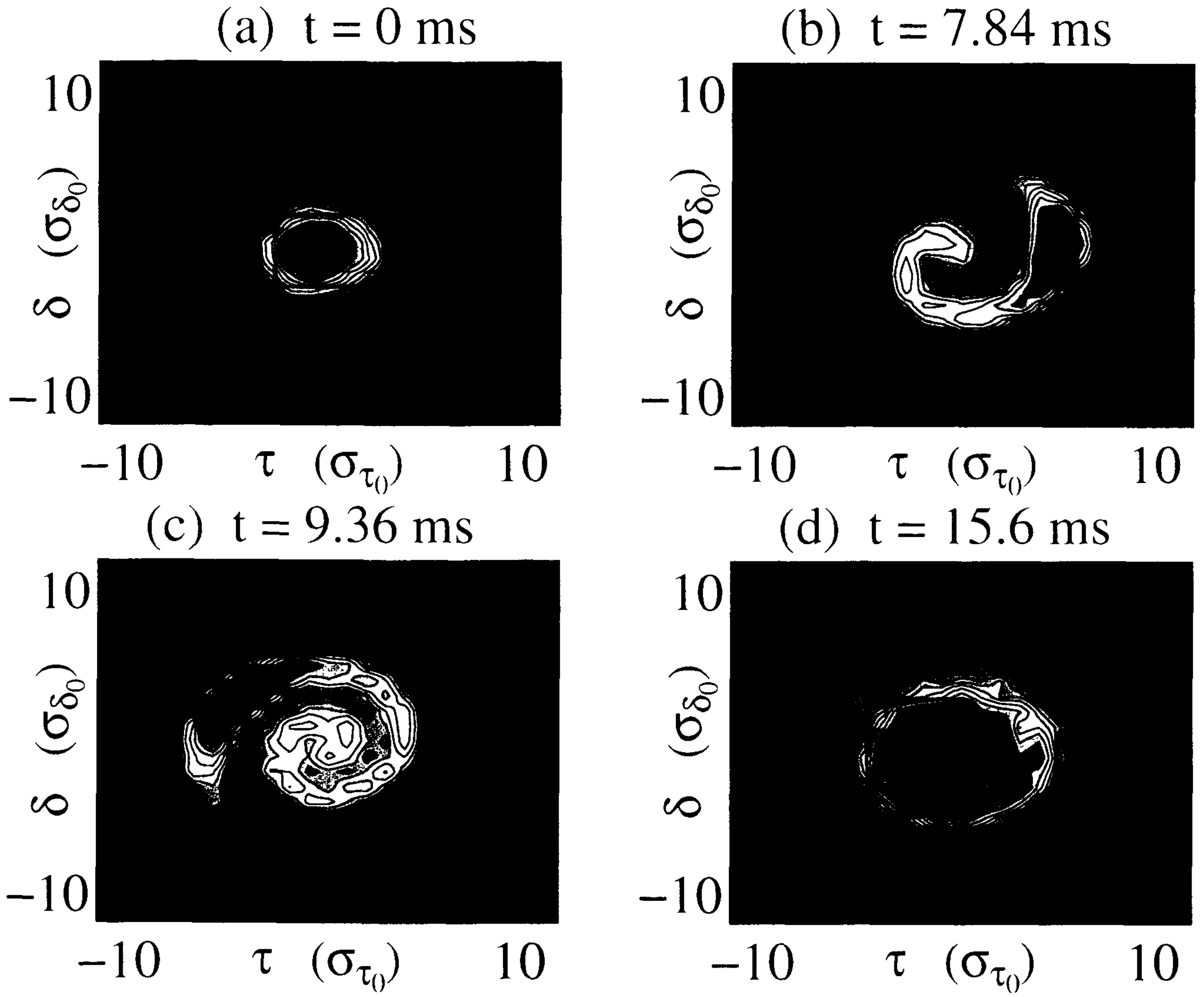
bunch dist. (ns)
-0.7
0.7
0
time (ms)
50



(a)
0.7
bunch dist. (ns)
-0.7
0
time (μs)
500
(b)
0.7
bunch dist. (ns)
-0.7
0
time (μs)
100
(c)
0.7
bunch dist. (ns)
-0.7
0
time (μs)
100
(d)
0.7
bunch dist. (ns)
-0.7
0
time (μs)
100

8 Simulations

- Photon intensity from optical port too low to get all the information desired from longitudinal distribution.
- Used a multiparticle tracking code to get more information about the behavior of the relaxation oscillation.
- Implements the standard synchrotron equations of motion, including quantum • uctuations and wake function.
- The long range wake is calculated from turn to turn using propagators.
- The code reproduced well the main behavior:
  - Instability thresholds
  - Relaxation oscillation and its frequency
  - Diffusion from bunch
- Predictions from simulations
  - Diffusion is from the front of the bunch
  - Particles spiral toward the center of the bunch to restart cycle


(a) t = 0 ms
(b) t = 7.84 ms
(c) t = 9.36 ms
(d) t = 15.6 ms
10
−10
$\delta\ (\sigma_{\delta_0})$
$\tau\ (\sigma_{\tau_0})$

9 Analytical Model

9.1 Goals

Based on the experimental data and computer simulations, we would like the analytical model to explain:

- Instability thresholds given by linear theory
- Saturation mechanism of oscillation
- Diffusion, from the head of the bunch, as the amplitude grows
- Conditions for relaxation oscillation
  - Formation of second attractor
  - Damping mechanism
- For $\omega_z > \omega_s$, behavior of second attractor
  - $\sim \pi$ out of phase with initial attractor
  - growth as initial attractor damps

9.2 Driving Term – Wake• eld

Our goal is to express the driving term of this oscillation, the wake• eld, in a closed, analytical form that can be exploited for calculations. We use the experimental data and vastly different time scales of the system to justify the following simplifying approximations:

- $(\sigma_\tau \ll 2\pi/\omega_R) \Rightarrow$ use impulse approximation for single pass of wake• eld $(Q \gg 1)$

$$V(t-u) = NeW(t-u) = NeU(t-u)\, 2\alpha_R R_S e^{-\alpha_R(t-u)} \cos\omega_R(t-u)$$

- (cavity length $\ll$ ring circumference) $\Rightarrow$ total wake is in• nite summation

$$V(t) = 2\alpha_R R_S (Ne) \sum_{k=-\infty}^{n} e^{-\alpha_R(t(nT_0)-u(kT_0))} \cos\omega_R(t(nT_0) - u(kT_0))$$

- $(\alpha_R \ll f_0) \Rightarrow$ approximate summation with integral

$$V(t) = 2\alpha_R R_S \left(\frac{Ne}{T_0}\right) \int_{-\infty}^{t} e^{-\alpha_R(t-u)} \cos\omega_R(t-u)\, du$$

- $t = nT_0 + \tau_t, u = kT_0 + \tau_u, \omega_R = p\omega_0 + \omega_z \Rightarrow$ remove multiples of 2π from cos

$$V(t) \cong 2\alpha_R R_S I \int_{-\infty}^{t} e^{-\alpha_R(t-u)} \cos[\omega_R(\tau_t - \tau_u) + \omega_z(t-u)]\, du$$

- instantaneously, motion always closely approximates harmonic oscillator
- $\alpha_R \sim \omega_s$ so memory of integral only lasts a few synchrotron periods
  $\Rightarrow \tau_t \cong r_t \cos\left(\omega_{st} n T_0 + \phi_t\right)$

$$V(t) = 2\alpha_R R_S I \int_{-\infty}^{t} e^{-\alpha_R(t-u)} \cos\left[\omega_R\left(r_t \cos\left(\omega_{st} t + \phi_t\right) - r_u \cos\left(\omega_{su} t + \phi_u\right)\right) + \omega_z (t-u)\right] du$$

The driving term is now analytically integrable, resulting in an in• nite Fourier-Bessel series

$$V(t) = 2\alpha_R R_S I \,\mathrm{Re}\left\{ \sum_{p,m=-\infty}^{\infty} \frac{j^{p-m} J_p(r_t) J_m(r_u) e^{j(p\omega_{st}+m\omega_{su})t} e^{jp\phi_t + jm\phi_u}}{\alpha_r + j\left(m\omega_{su} - \omega_z\right)} \right\}$$

This is still a very complex expression, so we exploit the slowly varying character, w.r.t. ω_s, of the amplitude, r_t, and phase, ϕ_t, of the oscillation.

9.3 Krylov-Bogoliubov-Mitropolskii (KBM) Averaging Method

- Driven harmonic oscillator

$$\ddot{\tau} + \omega_{s_0}^2 \tau = f_\tau(\tau, \dot{\tau})$$

- De• ne $r(t)$, $\phi(t)$, from modi• ed version of homogeneous solutions

$$\begin{aligned} \tau &= r(t)\cos(\omega_{s_0} t + \phi(t)) \\ \dot{\tau} &= -\omega_{s_0} r(t)\sin(\omega_{s_0} t + \phi(t)) \end{aligned}$$

- Solve the following equations

$$\begin{aligned} \frac{d\tau}{dt} &= \dot{\tau} \\ \frac{d\dot{\tau}}{dt} &= \ddot{\tau} = f_\tau(\tau, \dot{\tau}) - \omega_{s_0}^2 \tau \end{aligned}$$

- Obtain differential equations for $r(t)$, $\phi(t)$

$$\begin{aligned} \dot{r} &= -\frac{1}{\omega_{s_0}} \sin(\omega_{s_0} t + \phi) f(r, \phi) \\ \dot{\phi} &= -\frac{1}{r\omega_{s_0}} \cos(\omega_{s_0} t + \phi) f(r, \phi) \end{aligned}$$

- Average over one period (Fourier components)

$$\begin{aligned}\dot{\bar{r}} &= -\frac{1}{2\pi}\int_{t-\frac{2\pi}{\omega_{s_0}}}^{t} \sin\left(\omega_{s_0}\tau + \phi\right) f\left(r, \phi\right) d\tau \\ \dot{\bar{\phi}} &= -\frac{1}{2\pi r}\int_{t-\frac{2\pi}{\omega_{s_0}}}^{t} \cos\left(\omega_{s_0}\tau + \phi\right) f\left(r, \phi\right) d\tau\end{aligned}$$

- Equations of motion include terms from wake, radiation damping, and pendulum equation

$$\begin{aligned}\dot{\bar{r}}_t &= -\frac{1}{2\omega_{st}} F_{S1}\left(\bar{r}, \bar{\phi}\right) - \alpha_{rad}\bar{r}_t \\ \dot{\bar{\phi}}_t &= -\frac{1}{2\bar{r}_t\omega_{st}} F_{C1}\left(\bar{r}, \bar{\phi}\right) - \frac{1}{16}\bar{r}_t^2\omega_{st}\end{aligned}$$

where $F_{C1}\left(\bar{r}, \bar{\phi}\right)$ and $F_{S1}\left(\bar{r}, \bar{\phi}\right)$ are the Fourier coef• cients w.r.t. $\left(\omega_{s_0}\tau + \phi\right)$.

For a two particle model, the wake, and therefore its Fourier coef• cients, depend on the properties of both particles. These coef• cients, for a system with a source particle (r_u, ϕ_u) carrying current, I, acting on a test particle (r_t, ϕ_t) with in• nitesimal charge, are

$$F_{S1} = -A \sum_{k=1}^{\infty} J_k(r_u)\left[J_{k-1}(r_t) + J_{k+1}(r_t)\right] \times \left[\left(a_k^- - a_k^+\right)\cos(k\Delta\phi) - \left(b_k^- - b_k^+\right)\sin(k\Delta\phi)\right]$$

$$F_{C1} = A2b_0^+ J_0(r_u) J_1(r_t) + A \sum_{k=1}^{\infty} J_k(r_u)\left[J_{k-1}(r_t) - J_{k+1}(r_t)\right] \times \left[\left(b_k^- - b_k^+\right)\cos(k\Delta\phi) + \left(a_k^- - a_k^+\right)\sin(k\Delta\phi)\right]$$

where $A = 2I(\alpha_R R_S)\frac{\omega_{s0}^2}{V_{RF}|\cos\phi_s|}$, $\Delta\phi = \phi_t - \phi_u$and

$$a_k^{\pm} = \frac{\alpha_R}{\alpha_R^2 + (k\omega_{su} \pm \omega_z)^2}; \quad b_k^{\pm} = \frac{(k\omega_{su} \pm \omega_z)}{\alpha_R^2 + (k\omega_{su} \pm \omega_z)^2}$$

which are proportional to $\mathrm{Re}\{Z(k\omega_{su} \pm \omega_z)\}$ and $\mathrm{Im}\{Z(k\omega_{su} \pm \omega_z)\}$, respectively.

9.4 Analysis

Although this is still an in• nite sum, the sum depends on $\bar{r}$ the oscillation amplitude of the particles. From our data, this limits at about $\pi/2$, so the series converges quickly. For our parameters, in fact, a very good understanding comes from keeping only the lowest terms. These forces have a strong dipole characteristic.

9.4.1 Linear theory

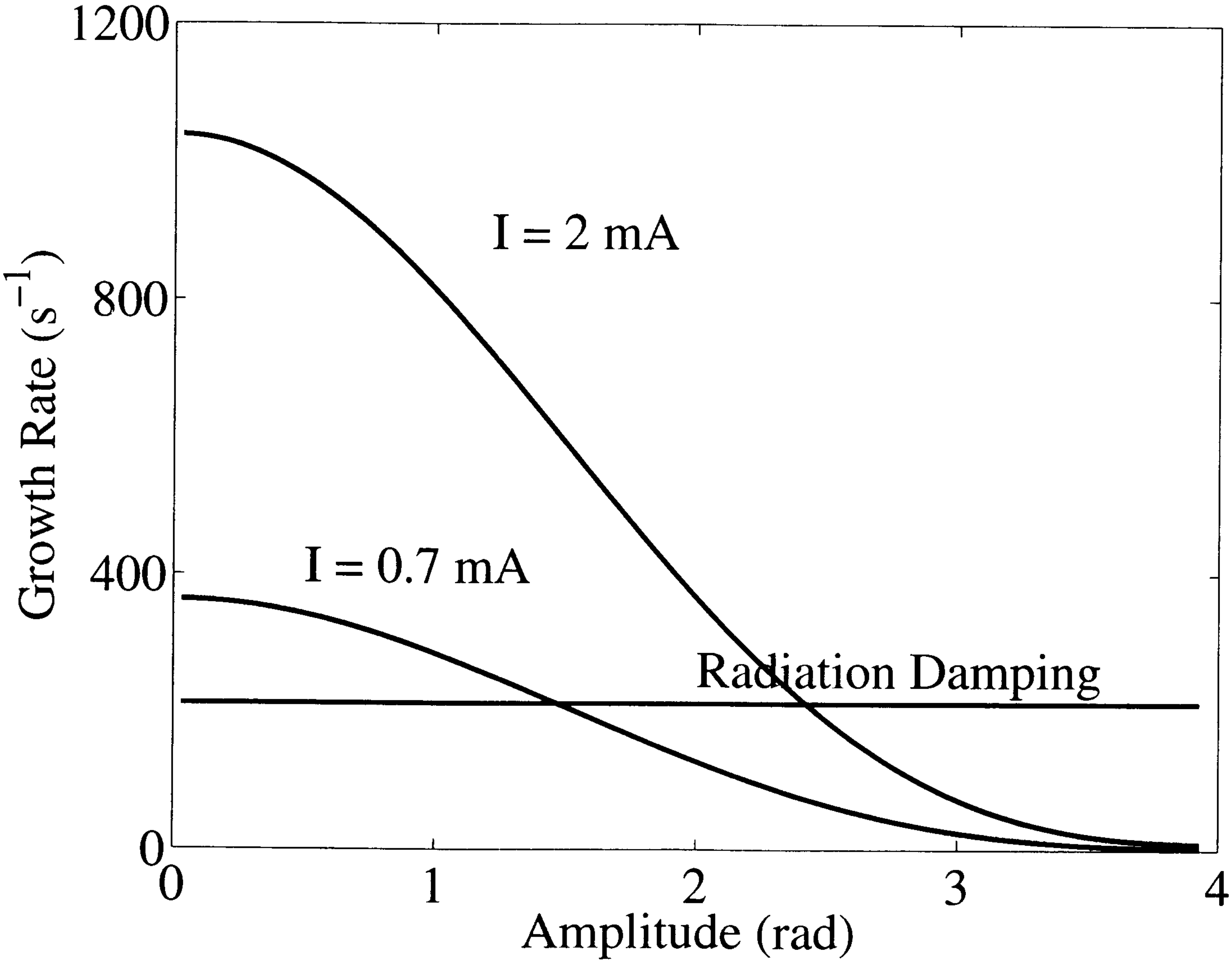
$(r_t, \phi_t) = (r_u, \phi_u)$ and use small amplitude expansion of $J_n(r)$.

- Gives instability thresholds of linear theory
- Shows odd symmetry of growth/damping and frequency shift with respect to the fractional part of the resonator frequency, ω_z

9.4.2 Saturation

$(r_t, \phi_t) = (r_u, \phi_u)$ but now evaluate non-linearities in sum

- Decrease in amplitude of sum with increasing argument gives saturation mechanism
- Data shows saturation at earlier level, consistent with particle loss from main bunch

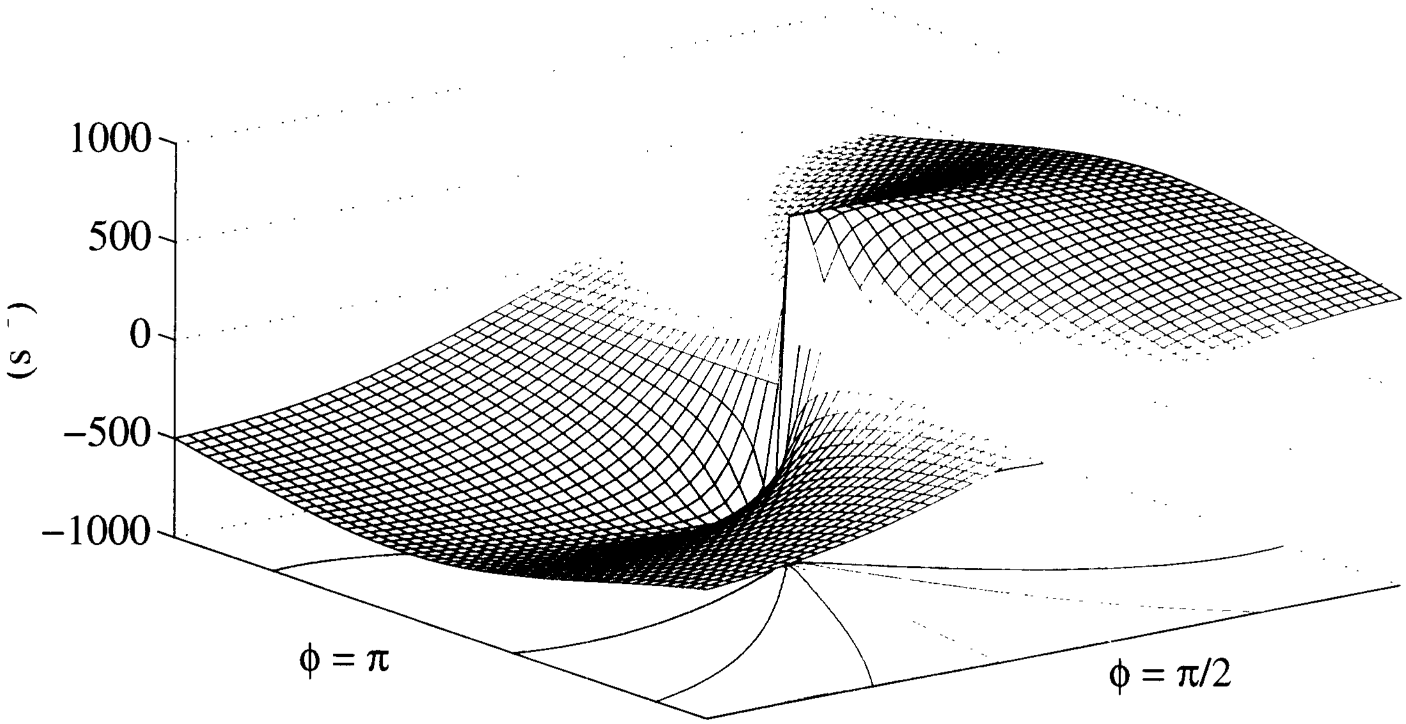


Growth Rate (s^{-1})
1200
800
400
0
I = 2 mA
I = 0.7 mA
Radiation Damping
0
1
2
3
4
Amplitude (rad)

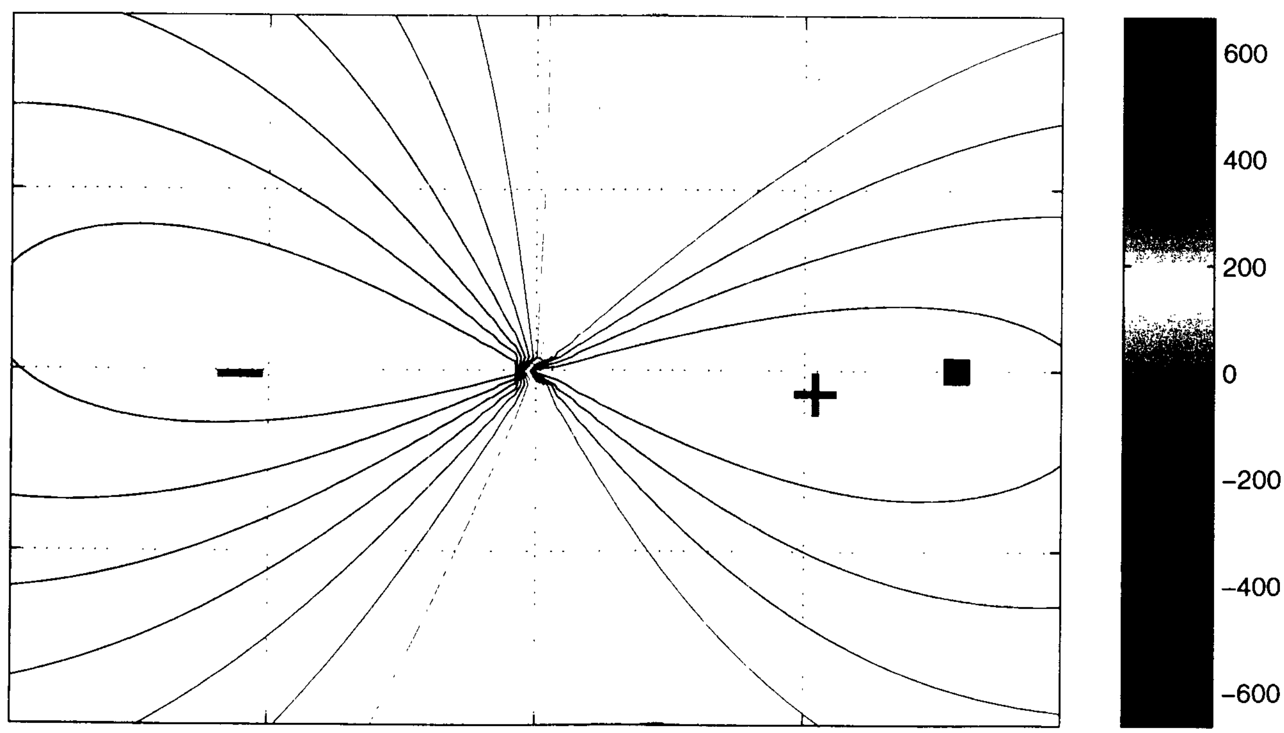
9.4.3 Diffusion

- To evaluate 'stability' of bunch as a macroparticle, look at situation of source particle, (r_u, ϕ_u), carrying all charge and examine behavior of test particle, (r_t, ϕ_t), at arbitrary location in phase space.
- Move to rotating coordinate frame in which $\phi_u = 0$
- In this frame, consider if separation between (r_t, ϕ_t) and (r_u, ϕ_u) is increasing or decreasing
- At small amplitudes
  - $F_{S1} \Rightarrow$ radial restoring force
  - $F_{C1} \Rightarrow$ azimuthal restoring force
  - $\Rightarrow$ macroparticle is stable

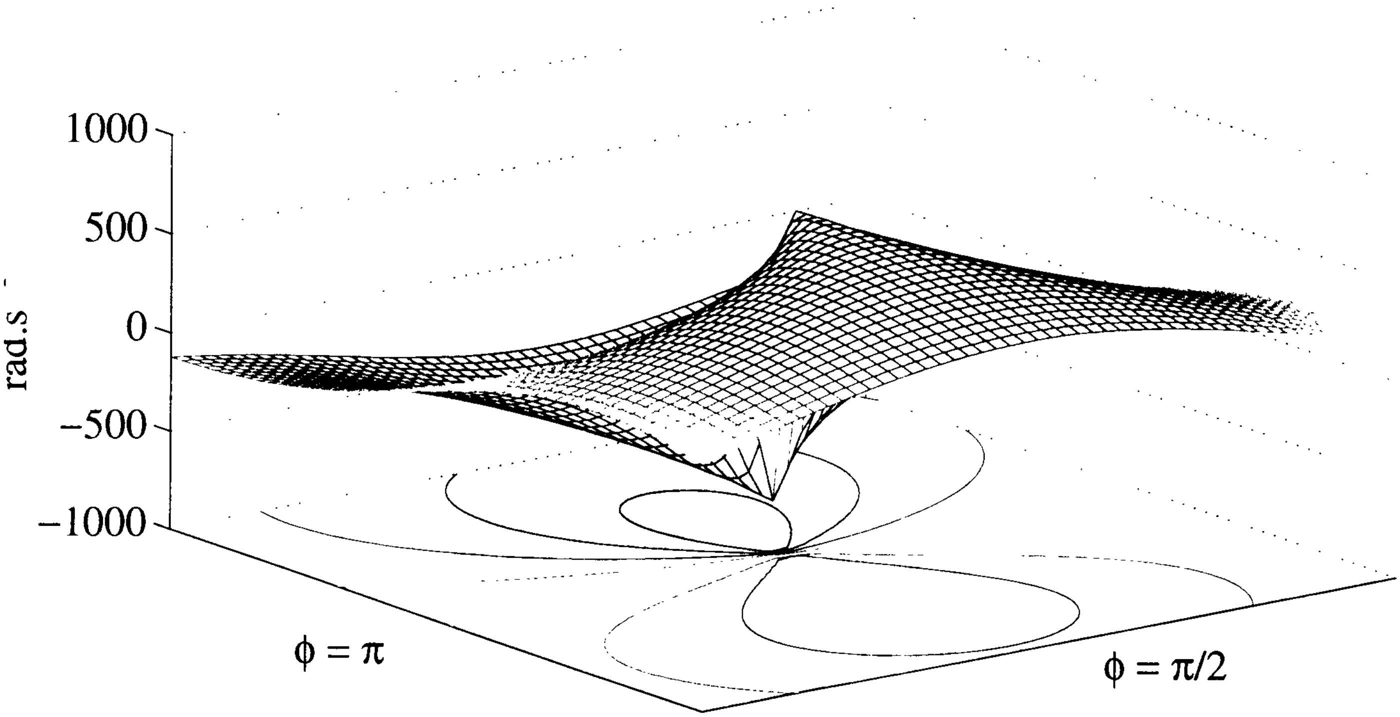
Growth from Wake



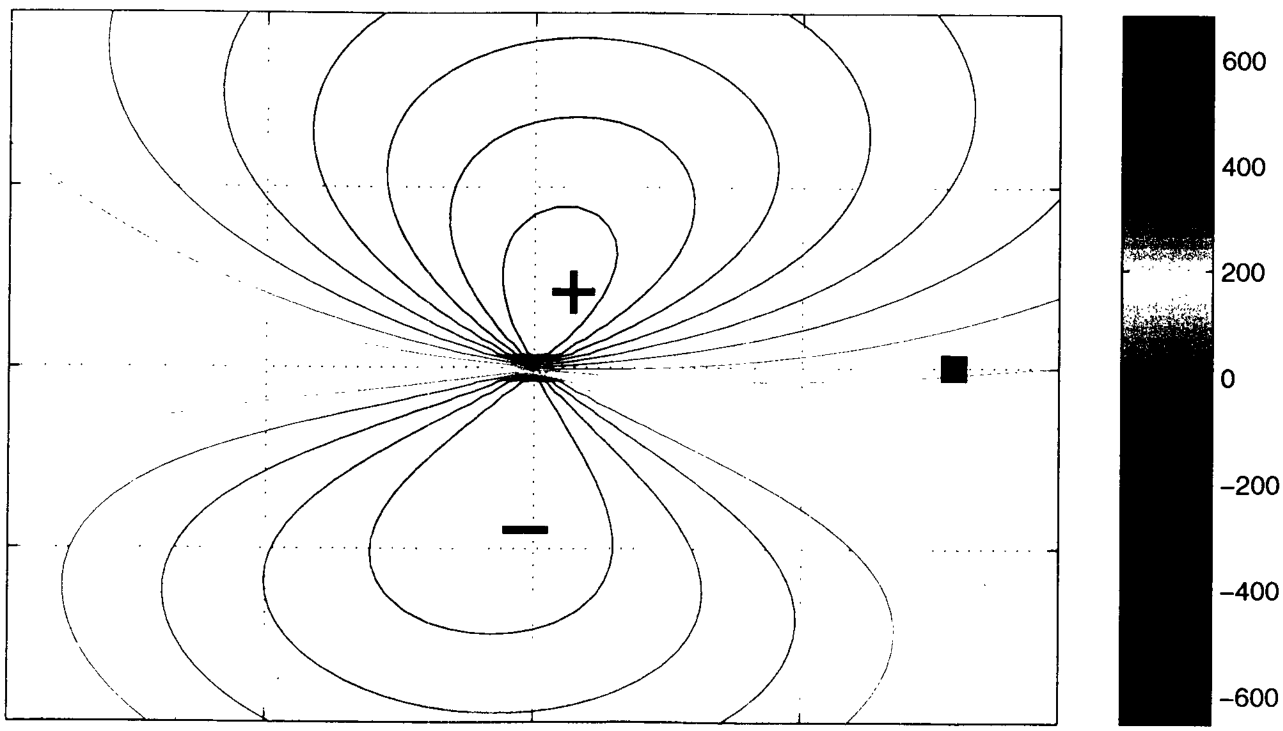
Main Body at π/4

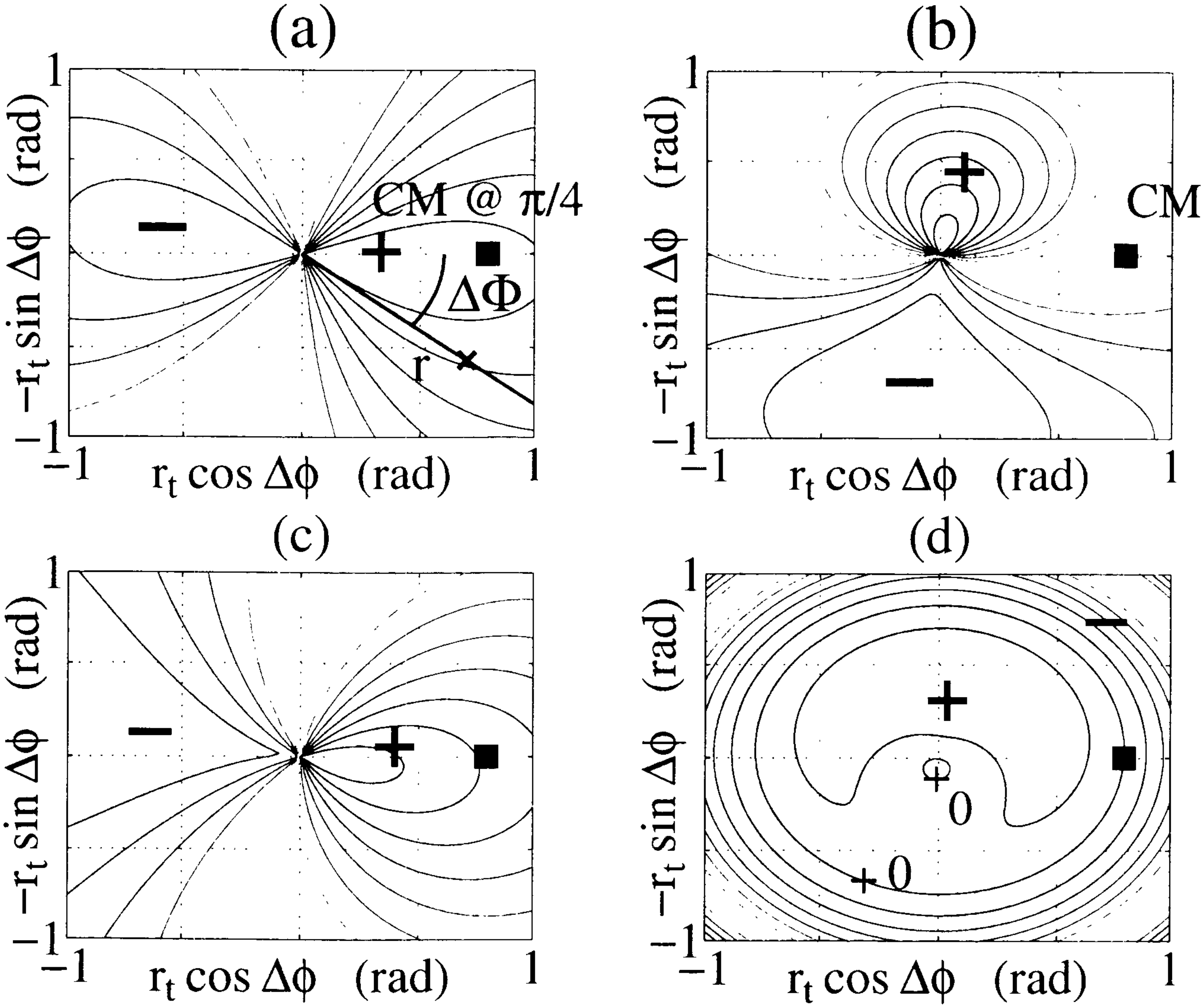


rdΦ/dt Frequency Shift from Wake



Main Body at $\pi/4$





(a)
CM @ π/4
ΔΦ
r
(b)
CM
(c)
(d)
0
0
r_t cos Δφ (rad)
−r_t sin Δφ (rad)

- At large amplitudes
  - Characteristics of radial forces are unchanged
  - Azimuthal force now dominated by pendulum effect
    * Test particle that falls behind macroparticle sees less $\bar{\dot{r}}$, drops toward the origin, gains $\bar{\dot{\phi}}$, and returns to region of larger $\bar{\dot{r}}$.
    * Test particle that moves ahead also sees less $\bar{\dot{r}}$, drops toward the origin, gains even more $\bar{\dot{\phi}}$, and exits macroparticle from the front.
  - As test particle rotates away from macroparticle, the net wake forces oscillate and lose their strength. Radiation damping brings the test particle back toward the origin.
  - Loss of charge from macroparticle also brings it closer to the origin

This change from attraction to the macroparticle to repulsion from it shows the change in dynamics of the system needed to form the second attractor and create the necessary conditions for the relaxation oscillation.

9.4.4 Location of second attractor near origin

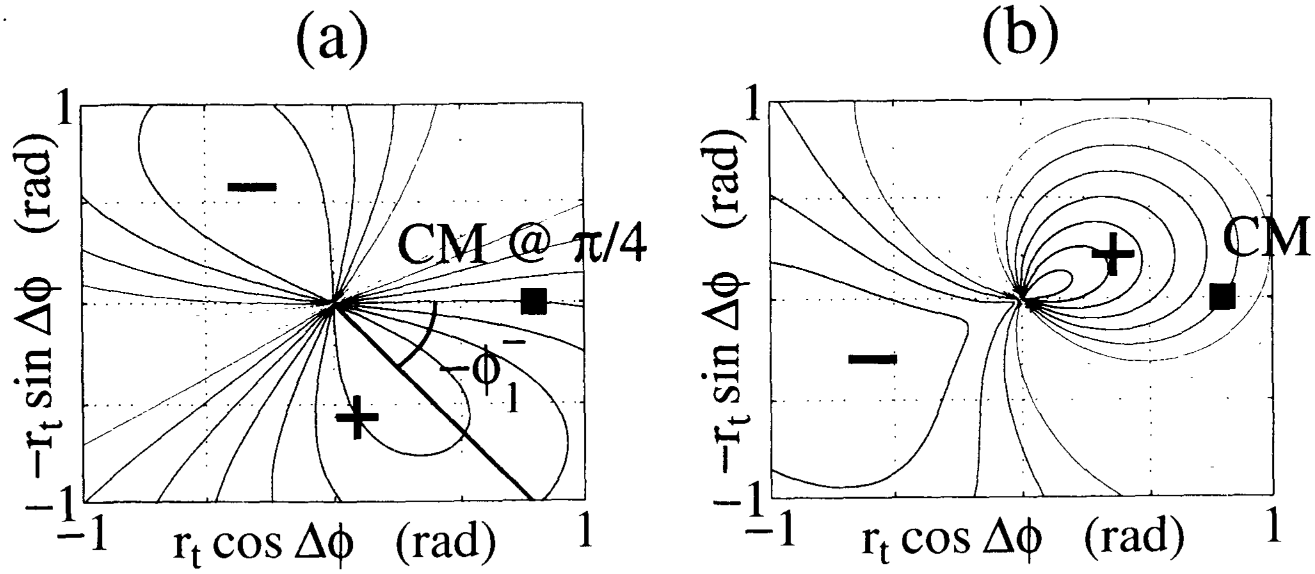
When the macroparticle has • nite amplitude, a test particle at the origin still sees • nite wake terms. Since the equation for $\dot{\bar{\phi}}$ has an additional r^{-1} factor multiplying the wake contribution, $\dot{\bar{\phi}}$ can assume any value near the origin. In particular, there exists a locus of points near the origin that are phase locked to the macroparticle. When the test particle reaches this locus, it again sees the wake of the macroparticle.

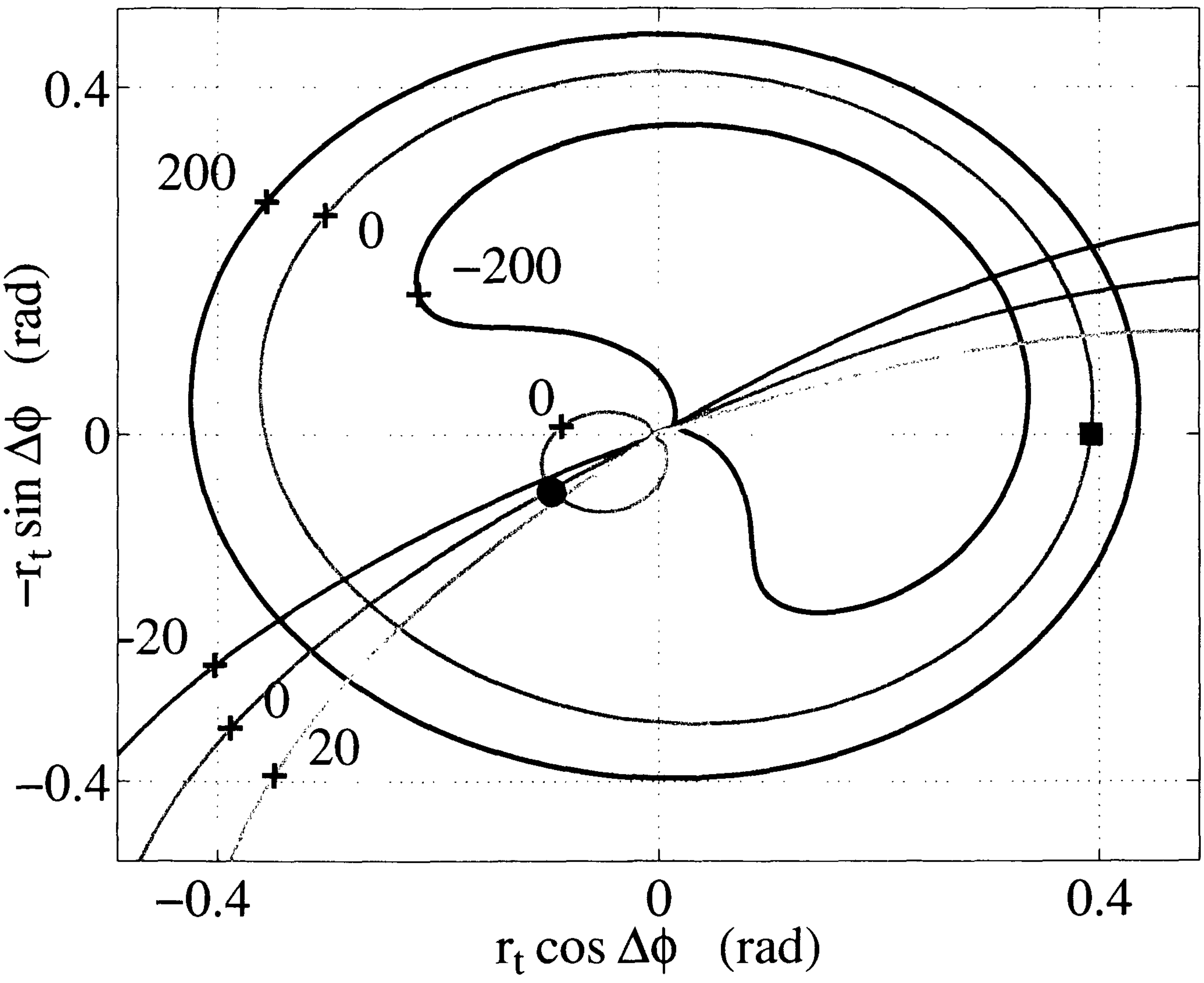
- Second attractor moves slowly $\Rightarrow$ close to • xed point $(\dot{\bar{r}} = 0)$ about $\pi/2$ ahead of macroparticle
- As particles accumulate at this second attractor its charge increases
  - Contributes to its own growth
  - Azimuthal coordinate increases since it needs to see damping from original attractor
  - Amplitude grows
  - Two centers exert damping force on each other
  - Initial attractor damps

Relaxation cycle is then • ow of particles between two centers, with cycle frequency $1/2$ that of observed signal.

9.4.5 Asymmetry of damping and observation of second attractor

- When $\omega_z > \omega_s$, F_{S1} and F_{C1} rotate clockwise with two results
- Test particles pass through higher regions of F_{S1} as they try to escape from the front of the bunch
  $\Rightarrow$ longer time needed for macroparticle to decay
- Node of shifted closer to $\pi \Rightarrow$ second attractor close to $\dot{r} = 0$ out of phase with initial attractor


(a)
(b)
CM @ π/4
CM
−r_t sin Δφ (rad)
r_t cos Δφ (rad)
−φ_1^−
1
−1
+
−



200
0
−200
0
−20
0
20
0.4
0
−0.4
−0.4
0
0.4
$-r_t \sin\Delta\phi$ (rad)
$r_t \cos\Delta\phi$ (rad)

10 Conclusions

- Experimental observations
  - Data gives complete characterization of phenomenon
- Simulations
  - Good agreement with experiments (low frequency behavior, • lamentation)
  - Additional predictive power of direction of • lamentation
- Analytic model
  - Exact derivation of long term wake potential
  - Theoretical explanation of phenomenon explains
    * Instability theory of linear model
    * Saturation mechanism of synchrotron oscillation
    * Diffusion mechanism
    * Conditions for relaxation oscillation
    * Creation and location of second attractor
    * Asymmetry of damping mechanism